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# Extension of the Unit Disk Gyrogroup into the Unit Ball of Any Real Inner Product Space

Abraham A. Ungar

*Department of Mathematics, North Dakota State University, Fargo, North Dakota 58105*



The group of all holomorphic automorphisms of the complex unit disk consists of Möbius transformations involving translation-like holomorphic automorphisms and rotations. The former are called *gyrotranslations*. As opposed to translations of the complex plane, which are associative-commutative operations (i.e., their composition law is associative and commutative) forming a group, gyrotranslations of the complex unit disk fail to form a group. Rather, left gyrotranslations are gyroassociative-gyrocommutative operations (i.e., their composition law is gyroassociative and gyrocommutative) forming a *gyrogroup*. The complex unit disk gyrogroup has previously been studied by the author (*Aequationes Math*. **47**, 1994, 240–254). Employing analogies shared by complex numbers and linear transformations of vector spaces, we extend in this article the complex disk gyrogroup and its Möbius transformations into the ball of any real inner product space and its generalized Möbius transformations. A gyrogroup is a mathematical object which first arose in the study of relativistic velocities which, under velocity addition, form a nongroup gyrogroup, as opposed to prerelativistic velocities, which form a group under velocity addition. It has been discovered that the mathematical regularity, seemingly lost in the transition from prerelativistic to relativistic velocities, is concealed in a relativistic effect known as Thomas precession. In its abstract context, Thomas precession is called Thomas gyration, giving rise to our "gyroterminology." Our gyroterminology, developed by the author (*Amer. J. Phys*. **59**, 1991, 824–834), involves terms like gyrogroups, gyroassociative-gyrocommutative laws, and gyroautomorphisms, in which we extensively use the prefix "gyro." 

## 1. INTRODUCTION

Employing analogies shared by complex numbers and linear transformations of vector spaces, we extend in this article the complex disk gyrogroup, studied in [19], into the ball of any real inner product space. A gyrogroup is a grouplike mathematical object which first emerged in the study of relativistically admissible velocities and their addition law [17]. It was emphasized in [17] that relativistic velocities form a nongroup groupoid under relativistic velocity addition, as opposed to prerelativistic velocities, which do form a group under ordinary vector addition. In particular, prerelativistic velocity addition is both associative and commutative while in general, as shown in [17], relativistic velocity addition is neither associative nor commutative. Consequently, a natural question arises as to whether indeed some mathematical regularity is lost in the progress from prerelativistic velocities to relativistic ones.

It was discovered in [17] that the seemingly lost mathematical regularity in the transition from prerelativistic to relativistic velocities is concealed in Thomas precession. Thomas precession is a relativistic space rotation which has no prerelativistic counterpart, and which was abstracted in [17] to the so-called *Thomas gyration*. To be more specific, it was discovered in [17] that the associative-commutative laws of prerelativistic velocity addition, lost in the transition to relativistic velocities, in fact reappear as gyroassociative-gyrocommutative laws of relativistic velocity addition. The gyroassociative-gyrocommutative laws involve the Thomas gyration and give rise to the grouplike mathematical object called a gyrogroup. It is obviously Thomas gyration which gives rise to our ‘‘gyroterminology’’ in which we extensively use the prefix ‘‘gyro.’’

In Section 2 we briefly describe the simplest known infinite gyrogroup, that is, the gyrogroup of the complex disk, which has been studied in [19]. It is this gyrogroup structure that we extend in Section 3 to balls in any real inner product space. In Section 4 we generalize the Poincaré metric of the complex disk into a metric of the ball in any real inner product space. In Section 5 we present some vectorlike features of the ball, arising from a scalar multiplication that we define between real numbers and elements of the ball. Related studies of balls in any complex inner product space are presented in [20–22]. The holomorphic automorphisms of the disk are Möbius transformations. Accordingly, the holomorphic automorphisms of the balls that we study are generalized Möbius transformations. Two invariants of the generalized Möbius transformations are presented in Section 6. One of these is the *scalar cross ratio* which is a straightforward generalization of the modulus of the cross ratio in the complex plane $\mathbb{C}$.

## 2. THE UNIT DISK GYROGROUP

A groupoid is a pair $(P, +)$ of a non-empty set $P$ with a binary operation $+$. An automorphism of a groupoid $(P, +)$ is a bijection of $P$ that respects its binary operation $+$. Following [19], a gyrogroup is a groupoid $(P, +)$ such that if we define the map gyr by the equation

$$\mathrm{gyr}[x; y]z = -(x + y) + (x + (y + z))$$

then, for all $x, y, z \in P$, $\mathrm{gyr}[x; y]$ is an automorphism of $(P, +)$,

$$\mathrm{gyr}\colon P \times P \to \mathrm{Aut}(P, +),$$

and

| | | |
|---|---|---|
| (G1) | $x + y \in P$ | closure property |
| (G2a) | $x + (y + z) = (x + y) + \mathrm{gyr}[x; y]z$ | right gyroassociative law |
| (G2b) | $(x + y) + z = x + (y + \mathrm{gyr}[y; x]z)$ | left gyroassociative law |
| (G3) | $x + y = \mathrm{gyr}[x; y](y + x)$ | gyrocommutative law |
| (G4) | $0 + x = x + 0 = x$ | existence of an identity element |
| (G5) | $(-x) + x = x + (-x) = 0$ | existence of inverse |
| (G6) | $\mathrm{gyr}[0; y] = I$ | identity gyroautomorphism |
| (G7) | $\mathrm{gyr}[x + y; y] = \mathrm{gyr}[x; y]$ | loop property. |

We have seen in [19] that the pair $(D_c, \oplus)$ consisting of (i) the open $c$-disk

$$D_c = \{x \in \mathbb{C}\colon |x| \geq c\}$$

having some positive radius $c$ in the complex plane $\mathbb{C}$, and (ii) the complex Einstein's addition law

$$x \oplus y = \frac{x + y}{1 + \bar{x}y/c^2}, \qquad x, y \in D_c, \tag{2.1}$$

forms a gyrogroup whose gyrooperation

$$\mathrm{gyr}\colon D_c \times D_c \to \mathrm{Aut}(D_c, \oplus)$$

is given by the equation

$$\mathrm{gyr}[x; y] = \frac{x \oplus y}{y \oplus x}. \tag{2.2}$$

The complex Einstein's addition law (2.1) is a Möbius transformation, the importance of which in the study of the holomorphic automorphism group of the complex disk is well known; see, e.g., [9, p. 3].

The grouplike nature of gyrogroups is exhibited in the existence of a unique solution in any gyrogroup to each of the two equations [19]

$$x \oplus a = b, \tag{2.3a}$$

whose solution is

$$x = b \ominus \mathrm{gyr}[b; a]a, \tag{2.3b}$$

and

$$a \oplus x = b, \tag{2.4a}$$

whose solution is

$$x = (-a) \oplus b, \tag{2.4b}$$

where we use the obvious notation $a \ominus b$ to denote $a \oplus (-b)$.

Suggestively, $a \oplus x$ ($x \oplus a$) is called a left (right) *gyrotranslation* of $x$ by $a$. Equations (2.3)–(2.4) thus demonstrate that gyrotranslations of $D_c$ are bijective.

By employing analogies shared by complex numbers and linear transformations of vector spaces we will extend in this article the gyrogroup $(D_c, \oplus)$ of the $c$-disk of a complex plane $\mathbb{C}$ into the gyrogroup $(V_c, \oplus)$ of the open $c$-ball $V_c$ of any real inner product space $V_\infty$. Accordingly, the holomorphic automorphisms of $D_c$, which are Möbius transformations, will be extended to holomorphic automorphisms of $V_c$, called generalized Möbius transformations.

Without loss of generality one may select $c = 1$. However, in general we refrain from taking advantage of the simplification offered by $c = 1$ in order to exhibit useful reductions which are obtained in the limit of large $c$ by letting $c \to \infty$. Thus, for instance, in the limit of large $c$ the noncommutative binary operation $\oplus$ in $D_c$ reduces to the commutative binary operation $+$ in $\mathbb{C}$, and Thomas gyration vanishes, that is, it reduces to the identity automorphism. Accordingly, the two equations (2.3a) and (2.4a) in the gyrogroup $(D_c, \oplus)$ reduce to the equation $a + x = b$ in $\mathbb{C}$, and their respective solutions in (2.3b) and (2.4b) reduce to $x = b - a$.

## 3. EXTENSION TO HOLOMORPHIC AUTOMORPHISMS OF THE BALL IN ANY REAL INNER PRODUCT SPACE

Complex numbers and linear transformations of vector spaces share remarkable analogies, enabling one to understand the latter in terms of the former [8]. Let $V_\infty = (V_\infty, +, \cdot)$ be a real inner product space with addition, $+$, and inner product, $\cdot$ [7]. We wish to extend the validity of the binary operation $\oplus$ from the disk $D_c$ in $\mathbb{C}$ in the open ball $V_c$ in $V_\infty$,

$$V_c = \{\mathbf{x} \in V_\infty : \|\mathbf{x}\| \geq c\},$$

by exploiting analogies shared by complex numbers and vectors. Here $\|\mathbf{x}\|$

is the norm of $\mathbf{x}$ induced by the inner product in $V_\infty$, $\|\mathbf{x}\| = \sqrt{\mathbf{x}^2} \geq 0$ where $\mathbf{x}^2 = \mathbf{x} \cdot \mathbf{x}$. Extensions of Möbius transformations and holomorphic mappings from the context of the complex plane to that of vector spaces and manifolds are discussed in the literature; see for instance [1, 3, 6, 11, 12]. Our extension is achieved by the viewing complex numbers $x = x_1 + ix_2$ in $\mathbb{C}$ as vectors, $\operatorname{vec}(x) = \mathbf{x} = (x_1, x_2)$, in $\mathbb{R}^2$. Once a complex identity is written as a 2-vector identity, an extension of the resulting vector identity to any abstract real inner product space becomes obvious. Three illustrative examples, all of which are needed for later reference, follow. In these examples a use is made of some cross product identities in arbitrary dimension [5].

Example I. A simple example is provided by the complex function $x\bar{y}$ where $x = x_1 + ix_2$ and $y = y_1 + iy_2$ are two complex numbers, and where $\bar{y}$ is the complex conjugate of $y$. Formally, we have

$$\begin{aligned} x\bar{y} &= x_1 y_1 + x_2 y_2 - i(x_1 y_2 - x_2 y_1) \\ &= \mathbf{x} \cdot \mathbf{y} - i\mathbf{x} \times \mathbf{y} \end{aligned}$$

where $\mathbf{x} = \operatorname{vec}(x)$ and $\mathbf{y} = \operatorname{vec}(y)$, so that

$$|1 + x\bar{y}|^2 = (1 + \mathbf{x} \cdot \mathbf{y})^2 + (\mathbf{x} \times \mathbf{y})^2 = (1 + \mathbf{x} \cdot \mathbf{y})^2 + \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - (\mathbf{x} \cdot \mathbf{y})^2. \tag{3.1a}$$

The cross product $\mathbf{x} \times \mathbf{y}$ is commonly defined in $\mathbb{R}^3$. However, the expression $(\mathbf{x} \times \mathbf{y})^2$ in Eq. (3.1a) is meaningful in any real inner product space if we borrow from the Euclidean 3-space $\mathbb{R}^3$ the identity

$$(\mathbf{x} \times \mathbf{y})^2 = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - (\mathbf{x} \cdot \mathbf{y})^2. \tag{3.1b}$$

The squared cross product in Eq. (3.1b) is given in terms of dot products and, hence, it is well defined in any inner product space. Equations (3.1) demonstrate that the real valued function $|1 + x\bar{y}|^2$ of the two complex variables $x$ and $y$ can be written as a real valued function of corresponding two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$. The later, in turn, remains meaningful in any real inner product space.

Example II. A second example is provided by the complex function $x\bar{y}z$ of $x, y, z \in \mathbb{C}$. Since vec: $\mathbb{C} \to \mathbb{R}^2$ takes $\mathbb{C}$ onto $\mathbb{R}^2$ according to the equation $\operatorname{vec}(x_1 + ix_2) = (x_1, x_2) = \mathbf{x}$, we have

$$\mathrm{vec}(x\bar{y}z) = (\mathbf{x}\cdot\mathbf{y})\mathbf{z} - (\mathbf{x}\times\mathbf{y})\times\mathbf{z}. \tag{3.2a}$$

To verify Eq. (3.2a) one shows that (i) $\mathrm{Re}(x\bar{y}z)$ equals the first component of the vector $(\mathbf{x}\cdot\mathbf{y})\mathbf{z} - (\mathbf{x}\times\mathbf{y})\times\mathbf{z}$ in $\mathbb{R}^2$; and that (ii) $\mathrm{Im}(x\bar{y}z)$ equals the second component in this vector. The right-hand side of Eq. (3.2a), which has been established in $\mathbb{R}^2$, is meaningful in any real inner product space if we borrow from $\mathbb{R}^3$ the identity

$$(\mathbf{x}\times\mathbf{y})\times\mathbf{z} = -(\mathbf{y}\cdot\mathbf{z})\mathbf{x} + (\mathbf{x}\cdot\mathbf{z})\mathbf{y} \tag{3.2b}$$

obtaining

$$\mathrm{vec}(x\bar{y}z) = (\mathbf{y}\cdot\mathbf{z})\mathbf{x} - (\mathbf{x}\cdot\mathbf{z})\mathbf{y} + (\mathbf{x}\cdot\mathbf{y})\mathbf{z}. \tag{3.2c}$$

EXAMPLE III. A slightly more involved example is provided by the complex valued function $(x\bar{y})^2 z$ of $x, y, z \in \mathbb{C}$,

$$\begin{aligned}\mathrm{vec}\{(x\bar{y})^2 z\} &= (\mathbf{x}\cdot\mathbf{y})^2\mathbf{z} - 2(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}\times\mathbf{y})\times\mathbf{z} \\ &\quad + 2(\mathbf{x}\times\mathbf{y})\times((\mathbf{x}\times\mathbf{y})\times\mathbf{z}) + (\mathbf{x}\times\mathbf{y})^2\mathbf{z}. \end{aligned} \tag{3.3a}$$

The right-hand side of Eq. (3.3a) is meaningful in any real inner product space if we borrow from $\mathbb{R}^3$ the identities (3.1b) and (3.2b) as well as the identity

$$\begin{aligned}(\mathbf{x}\times\mathbf{y})\times((\mathbf{x}\times\mathbf{y})\times\mathbf{z}) &= \{(\mathbf{x}\cdot\mathbf{y})(\mathbf{y}\cdot\mathbf{z}) - (\mathbf{x}\cdot\mathbf{z})\|\mathbf{y}\|^2\}\mathbf{x} \\ &\quad + \{(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}\cdot\mathbf{z}) - (\mathbf{y}\cdot\mathbf{z})\|\mathbf{x}\|^2\}\mathbf{y}. \end{aligned} \tag{3.3b}$$

A common feature of Examples I, II, and III is the careful use of the cross product $\mathbf{x}\times\mathbf{y}$ in an abstract real inner product space. While the vector produce $\mathbf{x}\times\mathbf{y}$ is commonly defined only in $\mathbb{R}^3$, the vector product expressions in Eqs. (3.1a), (3.2a), and (3.3a) are well defined in any real inner product space by the respective right-hand sides of Eqs. (3.1b), (3.2b), and (3.3b).

We are now in position to recognize that the complex-disk gyrogroup $D_c = (D_c, \oplus, \mathrm{gyr})$ can be interpreted as a gyrogroup $V_c = (V_c, \oplus, \mathrm{gyr})$ of the $c$-ball $V_c$ of any real inner product space $V_\infty$. Following Eq. (2.1) and Eqs. (3.1a) and (3.2a) the binary operation $\oplus$ in $D_{c=1}$ can be written as a binary operation in $\mathbb{R}^2_{c=1} = \{\mathbf{x}\in\mathbb{R}^2 : \|\mathbf{x}\| < c = 1\}$,

$$\begin{aligned}\operatorname{vec}(x \oplus y) &= \operatorname{vec}\left\{\frac{x+y}{1+\bar{x}y}\right\} \\ &= \operatorname{vec}\left\{\frac{1+x\bar{y}}{|1+\bar{x}y|^2}(x+y)\right\} \\ &= \frac{1+\mathbf{x}\cdot\mathbf{y}-(\mathbf{x}\times\mathbf{y})\times}{(1+\mathbf{x}\cdot\mathbf{y})^2+(\mathbf{x}\times\mathbf{y})^2}(\mathbf{x}+\mathbf{y}),\end{aligned} \tag{3.4}$$

where $x, y \in D_{c=1}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2_{c=1}$. Equation (3.4) presents a binary operation in the open $(c = 1)$-ball $\mathbb{R}^2_{c=1}$ of the real inner product space $\mathbb{R}^2$. It clearly suggests the binary operation $\oplus$ in the open unit ball of any real inner product space, $V_\infty$,

$$\mathbf{x} \oplus \mathbf{y} = \frac{1+\mathbf{x}\cdot\mathbf{y}-(\mathbf{x}\times\mathbf{y})\times}{(1+\mathbf{x}\cdot\mathbf{y})^2+(\mathbf{x}\times\mathbf{y})^2}(\mathbf{x}+\mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in V_{c=1}. \tag{3.5}$$

The vector product expressions $(\mathbf{x}\times\mathbf{y})\times(\mathbf{x}+\mathbf{y})$ and $(\mathbf{x}\times\mathbf{y})^2$ which appear in Eq. (3.5) are interpreted in $V_\infty$ by means of Eqs. (3.2b) and (3.1b). In view of Eq. (3.4), the transformation taking $\mathbf{z} \in V_c$ into $\mathbf{a} \oplus \mathbf{z} \in V_c$ is an extended Möbius transformation (other ways of extending Möbius transformations are known in the literature; see, e.g., [1, 3, 6, 11, 12]).

The gyrooperation gyr of the gyrogroup $V_{c=1} = (V_{c=1}, \oplus, \mathrm{gyr})$ is given by Eq. (2.2) with its interpretation in $V_\infty$ by means of Eqs. (3.1), (3.2), and (3.3) as

$$\begin{aligned}\operatorname{gyr}[\mathbf{x};\mathbf{y}]\mathbf{z} &= \operatorname{vec}\{\operatorname{gyr}[x;y]z\} = \operatorname{vec}\left\{\frac{x\oplus y}{y\oplus x}z\right\} = \operatorname{vec}\left\{\frac{1+x\bar{y}}{1+\bar{x}y}z\right\} \\ &= \operatorname{vec}\left\{\frac{(1+x\bar{y})^2}{|1+\bar{x}y|^2}z\right\} = \operatorname{vec}\left\{\frac{z+2x\bar{y}z+(x\bar{y})^2z}{|1+\bar{x}y|^2}\right\} \\ &= \frac{\operatorname{vec}(z)+2\operatorname{vec}(x\bar{y}z)+\operatorname{vec}\left((x\bar{y})^2z\right)}{|1+\bar{x}y|^2} \\ &= \frac{\begin{array}{r}\mathbf{z}+2(\mathbf{x}\cdot\mathbf{y})\mathbf{z}-2(\mathbf{x}\times\mathbf{y})\times\mathbf{z}+(\mathbf{x}\cdot\mathbf{y})^2\mathbf{z}-2(\mathbf{x}\cdot\mathbf{y})(\mathbf{x}\times\mathbf{y})\times\mathbf{z}\\ +2(\mathbf{x}\times\mathbf{y})\times((\mathbf{x}\times\mathbf{y})\times\mathbf{z})+(\mathbf{x}\times\mathbf{y})^2\mathbf{z}\end{array}}{(1+\mathbf{x}\cdot\mathbf{y})^2+(\mathbf{x}\times\mathbf{y})^2} \\ &= \left\{I+2\frac{-(1+\mathbf{x}\cdot\mathbf{y})+(\mathbf{x}\times\mathbf{y})\times}{(1+\mathbf{x}\times\mathbf{y})^2+(\mathbf{x}\times\mathbf{y})^2}(\mathbf{x}\times\mathbf{y})\times\right\}\mathbf{z},\end{aligned} \tag{3.6}$$

where $I$ is the identity automorphism of $V_{c=1}$. The extreme right-hand side of Eq. (3.6) involves two successive applications of the transformation $(\mathbf{x} \times \mathbf{y}) \times$ to $\mathbf{z}$. One should calculate these two successive applications to $\mathbf{z}$ as $(\mathbf{x} \times \mathbf{y}) \times ((\mathbf{x} \times \mathbf{y}) \times \mathbf{z})$ rather than $((\mathbf{x} \times \mathbf{y}) \times (\mathbf{x} \times \mathbf{y})) \times \mathbf{z}$; the latter gives $\mathbf{0}$ in $\mathbb{R}^3$ and is not extendible to abstract real inner product spaces. Equation (3.1a) that has been employed in the derivation of (3.6) is valid in $\mathbb{R}^2$. We are now going to extend the validity of (3.6) to any real inner product space by *definition*.

Following (3.6) we define the gyrooperation gyr: $V_{c=1} \times V_{c=1} \to \mathrm{Aut}(V_{c=1}, \oplus)$ as

$$\mathrm{gyr}[\mathbf{x}; \mathbf{y}] = I + 2\frac{-(1 + \mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \times \mathbf{y}) \times}{(1 + \mathbf{x} \cdot \mathbf{y})^2 + (\mathbf{x} \times \mathbf{y})^2}(\mathbf{x} \times \mathbf{y}) \times \tag{3.7}$$

for any $\mathbf{x}, \mathbf{y} \in V_{c=1}$, where the vector product expressions are interpreted in $V_\infty$ by means of Eqs. (3.1b), (3.2b), and (3.3b).

By employing analogies shared by the field of complex numbers and vector spaces we have formally generalized (i) the gyrogroup of all holomorphic automorphisms $z \mapsto a \oplus z$, called gyrotranslations, of the complex disk $D_c$ to (ii) the gyrogroup of all gyrotranslations $z \mapsto a \oplus z$ of the open ball $V_c$ of any real inner product space $V_\infty$. We summarize this in the following theorem which, for the special case of $V_\infty = \mathbb{R}^3$, has been presented in [16].

THEOREM 3.1. *The open ball $V_c$ of any real inner product space $V_\infty = (V_\infty, +, \cdot)$ possesses a gyrogroup structure, $V_c = (V_c, \oplus, \mathrm{gyr})$, with binary operation $\oplus$ and gyrooperation gyr given by the equations*

$$\mathbf{x} \oplus \mathbf{y} = \frac{1 + c^{-2}\mathbf{x} \cdot \mathbf{y} - c^{-2}(\mathbf{x} \times \mathbf{y}) \times}{(1 + c^{-2}\mathbf{x} \cdot \mathbf{y})^2 + c^{-4}(\mathbf{x} \times \mathbf{y})^2}(\mathbf{x} + \mathbf{y}) \tag{3.8}$$

*and*

$$\mathrm{gyr}[\mathbf{x}; \mathbf{y}] = I + \frac{2}{c^2}\frac{-(1 + c^{-2}\mathbf{x} \cdot \mathbf{y}) + c^{-2}(\mathbf{x} \times \mathbf{y}) \times}{(1 + c^{-2}\mathbf{x} \cdot \mathbf{y})^2 + c^{-4}(\mathbf{x} \times \mathbf{y})^2}(\mathbf{x} \times \mathbf{y}) \times \tag{3.9}$$

*for any $\mathbf{x}, \mathbf{y} \in V_c$.*

Theorem 3.1 can straightforwardly be verified by computer algebra. This theorem was obtained by employing the transformation vec: $\mathbb{C} \to \mathbb{R}^2$ to translate results in $\mathbb{C}$ into results in $\mathbb{R}^2$. Results in $\mathbb{R}^2$, then, were extended to the abstract real inner product space $V_\infty$. We should emphasize that care must be taken when extending a vector identity from $\mathbb{R}^2$ into higher dimensional vector spaces if the identity involves more than two vector indeterminates. Thus, for instance, we have in $\mathbb{R}^2$ the identity

$$\begin{aligned}((\mathbf{y}\times\mathbf{z})\cdot(\mathbf{v}\times\mathbf{z}))\mathbf{x}-((\mathbf{x}\times\mathbf{z})\cdot(\mathbf{v}\times\mathbf{z}))\mathbf{y}\\ +((\mathbf{x}\times\mathbf{y})\cdot(\mathbf{v}\times\mathbf{z}))\mathbf{z}=0\end{aligned} \tag{3.10a}$$

for any vectors $\mathbf{v},\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{R}^2$. This identity is valid in $\mathbb{R}^2$ due to a peculiarity of this space according to which any three vectors are linearly dependent. The left-hand side of Eq. (3.10a) is meaningful in any real inner product space if we borrow from $\mathbb{R}^3$ the identity

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}). \tag{3.10b}$$

One should not expect identity (3.10a) to remain valid in higher dimensional vector spaces where three linearly independent vectors do exist.

If we adopt the notation

$$\mathrm{vec}\left\{\frac{x+y}{1+c^{-2}\bar{x}y}\right\}=\frac{\mathbf{x}+\mathbf{y}}{1+c^{-2}\bar{\mathbf{x}}\mathbf{y}} \tag{3.11}$$

and

$$\mathrm{vec}\left\{\frac{x\oplus y}{y\oplus x}z\right\}=\frac{\mathbf{x}\oplus\mathbf{y}}{\mathbf{y}\oplus\mathbf{x}}\mathbf{z}, \tag{3.12}$$

then the equations in Theorem 3.1 can be written as

$$\mathbf{x}\oplus\mathbf{y}=\frac{\mathbf{x}+\mathbf{y}}{1+c^{-2}\bar{\mathbf{x}}\mathbf{y}} \tag{3.13}$$

and

$$\mathrm{gyr}[\mathbf{x};\mathbf{y}]=\frac{\mathbf{x}\oplus\mathbf{y}}{\mathbf{y}\oplus\mathbf{x}}, \tag{3.14}$$

as we see from Eqs. (3.4) and (3.6).

It follows from Eqs. (3.13) that our generalized *pure* Möbius transformations (that is, Möbius transformations *without rotation*; we may remark here that in a similar way it is customary in special theory of relativity to call a Lorentz transformation without rotation a *pure Lorentz transformation*) of the ball $V_c$ are, for all $\mathbf{x}\in V_c$, the bijective maps $\mathbf{y}\mapsto\mathbf{x}\oplus\mathbf{y}$ that we consider as a binary operation in $V_c$ given by the equation

$$\begin{aligned}\mathbf{x} \oplus \mathbf{y} &= \frac{\mathbf{x} + \mathbf{y}}{1 + c^{-2}\bar{\mathbf{x}}\mathbf{y}} \\ &= \frac{1 + c^{-2}\mathbf{x}\cdot\mathbf{y} - c^{-2}(\mathbf{x}\times\mathbf{y})\times}{(1 + c^{-2}\mathbf{x}\cdot\mathbf{y})^2 + c^{-4}(\mathbf{x}\times\mathbf{y})^2}(\mathbf{x} + \mathbf{y}) \\ &= \frac{1}{F^2(\mathbf{x},\mathbf{y})}\left\{(1 + 2c^{-2}\mathbf{x}\cdot\mathbf{y} + c^{-2}\mathbf{y}^2)\mathbf{x} + (1 - c^{-2}\mathbf{x}^2)\mathbf{y}\right\}, \qquad (3.15)\end{aligned}$$

where $F^2(\mathbf{x},\mathbf{y}) = (F(\mathbf{x},\mathbf{y}))^2$, $F(\mathbf{x},\mathbf{y}) > 0$, and

$$F^2(\mathbf{x},\mathbf{y}) = 1 + 2c^{-2}\mathbf{x}\cdot\mathbf{y} + c^{-4}\mathbf{x}^2\mathbf{y}^2 \qquad (3.16)$$

for all $\mathbf{x},\mathbf{y} \in V_c$.

The restriction $\mathbf{x},\mathbf{y} \in V_c \subset V_\infty$ ensures that $F^2(\mathbf{x},\mathbf{y})$ is positive so that $x \oplus y$ in (3.15) is non-singular and real. Clearly, $F(\mathbf{x},\mathbf{y})$ is defined by (3.16) for all $\mathbf{x},\mathbf{y} \in V_\infty$ where it satisfies the inequality $F^2(\mathbf{x},\mathbf{y}) \geq 0$, and $F(\mathbf{x},\mathbf{y}) = 0$ if and only if $\mathbf{x}$ and $\mathbf{y}$ (i) are antiparallell ($\mathbf{x} = -\lambda\mathbf{y}$ for some $\lambda > 0$, and $\mathbf{y} \neq \mathbf{0}$), and (ii) have magnitudes satisfying $\|\mathbf{x}\|\,\|\mathbf{y}\| = c^2$. Hence, the binary operation $\mathbf{x} \oplus \mathbf{y}$ is real or $\infty$ in $V_\infty$. By allowing singularities, we may thus view the gyrotranslations $\mathbf{z} \mapsto \mathbf{a} \oplus \mathbf{z}$ of $z$ as generalized pure Möbius transformations of $V_\infty$ into $V_\infty \cup \infty$ for all $a \in V_\infty$. This, in turn, suggests the following

DEFINITION 3.1 (Generalized Möbius Transformations). The map $\mathbf{z} \mapsto \mathbf{a} \oplus \mathbf{z}$ of $V_\infty$ into $V_\infty \cup \infty$ is called a *generalized pure Möbius transformation*. The set of all generalized Möbius transformations of $V_\infty$ into $V_\infty \cup \infty$ is the group of all generalized pure Möbius transformations of $V_\infty$ into $V_\infty \cup \infty$ and all rotations of $V_\infty$ about its origin.

Equation (3.13) indicates that (i) the function $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x} \ominus \mathbf{y}\|$ is a legitimate distance function in $V_c$ preserved by the gyrotranslations and rotations of $V_c$ (this will be verified in Section 4), and Eq. (3.14) indicates that (ii) the gyroautomorphisms $\mathrm{gyr}[\mathbf{x};\mathbf{y}]$ of $V_\infty$ (which are automorphisms of $V_c$ as well) preserve the norm in $V_\infty$, and that (iii) the inverse of $\mathrm{gyr}[\mathbf{x},\mathbf{y}]$ is $\mathrm{gyr}[\mathbf{y};\mathbf{x}]$. Both (ii) and (iii) can readily be verified by computer algebra.

The function $F(\mathbf{x},\mathbf{y})$ is a symmetric form in $V_c$, called a *cocycle form* [14], satisfying the *cocycle identity*

$$F(\mathbf{x},\mathbf{y} \oplus \mathbf{z})F(\mathbf{y},\mathbf{z}) = F(\mathbf{y} \oplus \mathbf{x},\mathbf{z})F(\mathbf{x},\mathbf{y}). \qquad (3.17)$$

The cocycle identity is known to be useful in various branches of mathematics; for a recent article about the cocycle equation and its relevance see [4]. Cocycle equations that appear in the literature involve a *group* binary operation. It is therefore interesting to realize that the cocycle identity in (3.17) involves a *gyrogroup* binary operation, $\oplus$. The fact that the gyrogroup $(V_c, \oplus)$ is equipped with a cocycle form, $F(\mathbf{x}, \mathbf{y})$, is crucial in the extension of the gyrogroup $(V_c, \oplus)$ into a corresponding Lorentz group, as shown in [4].

Theorem 3.1 and analogies with the complex disk $D_c$ suggest the following two definitions.

DEFINITION 3.2 (The Gyrogroup Induced by a Real Inner Product Space). Let $V_\infty = (V_\infty, +, \cdot)$ be a real inner product space and let $V_c$ be its open $c$-ball, $c$ being an arbitrarily fixed positive constant. The gyrogroup $V_c = (V_c, \oplus, \mathrm{gyr})$, whose gyrogroup operation $\oplus$ and gyrogroup gyrooperation gyr are defined in Eqs. (3.8) and (3.9), is called the gyrogroup induced by the space $V_\infty$ on its open $c$-ball $V_c$.

DEFINITION 3.3 (Holomorphic Automorphisms). Let $\mathrm{Aut}(V_\infty) = \mathrm{Aut}(V_\infty, +, \cdot)$ be the group of all automorphisms of the real inner product space $V_\infty$. The holomorphic automorphisms of the ball $V_c \subset V_\infty$ are the elements $(\mathbf{a}, A)$ of the gyrosemidirect product group $V_c \times_g \mathrm{Aut}(V_\infty)$ viewed as bijections of $V_c$, $(\mathbf{a}, A): V_c \to V_c$, given by the equation

$$(\mathbf{a}, A)\mathbf{x} = \mathbf{a} \oplus A\mathbf{x}$$

for any $(\mathbf{a}, A) \in V_c \times_g \mathrm{Aut}(V_\infty)$ and $\mathbf{x} \in V_c$. The holomorphic automorphisms $(\mathbf{a}, I)$ are called (left) gyrotranslations of $V_c$, and the holomorphic automorphisms $(0, A)$ are called rotations of $V_c$, where $\mathbf{0}$ is the zero element of $V_c$, and $I$ is the identity automorphism of $V_c$.

The gyrogroup structure of $V_c$, established in Theorem 3.1, will allow us in the next section to both (i) establish the (generalized) Poincaré metric, defined in Eq. (4.1); and (ii) present the holomorphic automorphism composition law, in Eq. (4.6).

## 4. THE (GENERALIZED) POINCARÉ METRIC OF $V_c$

We introduce the following definition of a generalized Poincaré distance function on the open $c$-ball $V_c$ in any real inner product space $V_\infty$.

DEFINITION 4.1. Let $V_c = (V_c, \oplus, \mathrm{gyr})$ be the gyrogroup induced by a real inner product space $V_\infty = (V_\infty, +, \cdot)$ on its open $c$-ball $V_c$. The (generalized) Poincaré distance function on $V_c$ is given by the equation

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \ominus \mathbf{y}\|. \tag{4.1}$$

Under certain circumstances, our generalized Poincaré distance function (4.1) reduces to the Poincaré distance function which is well known in the literature; see, e.g., [9, p. 29].

The binary operation $\oplus$ in the real interval $(-c, c)$ is applicable to the real numbers $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ where $\mathbf{x}, \mathbf{y} \in V_c$,

$$\|\mathbf{x}\| \oplus \|\mathbf{y}\| = \frac{\|\mathbf{x}\| + \|\mathbf{y}\|}{1 + c^{-2}\|\mathbf{x}\|\,\|\mathbf{y}\|},$$

enabling us to write the inequalities

$$\|\mathbf{x} \oplus \mathbf{y}\| \le \|\mathbf{x}\| \oplus \|\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\| \tag{4.2}$$

that will be verified in Theorems 4.1 and 4.2 below. We should note that in (4.2) the operation $\oplus$ denotes simultaneously a binary operation in the ball $V_c$ and in the interval $(-c, c)$.

THEOREM 4.1 (The Triangle Inequality for $V_c$). *For any* $\mathbf{x}, \mathbf{y} \in V_c$,

$$\|\mathbf{x} \oplus \mathbf{y}\| \le \|\mathbf{x}\| \oplus \|\mathbf{y}\|. \tag{4.3}$$

*Proof.* Let $\gamma_x = (1 - c^{-2}\|\mathbf{x}\|^2)^{-1/2}$ be the Lorentz factor of $\mathbf{x} \in V_c$. Clearly, $\gamma_x = \gamma_{\|\mathbf{x}\|}$. Following the formalism in Eq. (3.1a) and the identity in [16, Eq. 4.4] we have

$$\begin{aligned}
\gamma_{x\oplus y} &= \gamma_x\gamma_y\sqrt{(1 + \mathbf{x}\cdot\mathbf{y})^2 + (\mathbf{x}\times\mathbf{y})^2} \\
&= \gamma_x\gamma_y\sqrt{1 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{x}\|^2\|\mathbf{y}\|^2}\,.
\end{aligned}$$

Hence, by Cauchy−Schwartz inequality,

$$\begin{aligned}
\gamma_{\|\mathbf{x}\|\oplus\|\mathbf{y}\|} &= \gamma_{\|\mathbf{x}\|}\gamma_{\|\mathbf{y}\|}(1 + \|\mathbf{x}\|\,\|\mathbf{y}\|) \\
&= \gamma_x\gamma_y(1 + \|\mathbf{x}\|\,\|\mathbf{y}\|) \\
&\ge \gamma_x\gamma_y\sqrt{1 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{x}\|^2\|\mathbf{y}\|^2} \\
&= \gamma_{x\oplus y} \\
&= \gamma_{\|x\oplus y\|}.
\end{aligned}$$

But $\gamma_x = \gamma_{\|\mathbf{x}\|}$ is a monotonic increasing function of $\|\mathbf{x}\|$. Hence,

$$\|\mathbf{x} \oplus \mathbf{y}\| \le \|\mathbf{x}\| \oplus \|\mathbf{y}\|$$

as desired.

THEOREM 4.2. *Let* $a, b \in [0, c)$ *be two real numbers in the interval* $[0, c)$, $c > 0$. *Then*

$$a \oplus b \le a + b.$$

*Proof.* This follows from $a \oplus b = (a+b)/(1+ab/c^2)$.

The following lemma is interesting since it demonstrates applications of gyrogroup properties studied in [15] in the context of the abstract gyrogroup.

LEMMA 4.1. *Let $a, b, c$ be any three elements of a gyrogroup $P = (P, +, \mathrm{gyr})$. Then*

$$(a+b) - (a+c) = \mathrm{gyr}[a; b](b-c).$$

The proof of Lemma 4.1 is given in [19].

With the aid of Lemma 4.1 we show in the following theorem that the Poincaré distance function on $V_c$ is legitimate.

THEOREM 4.3 (The Triangle Inequality for the [Generalized] Poincaré Distance Function). *Let $V_c = (V_c, \oplus, \mathrm{gyr})$ be the gyrogroup induced by a real inner product space $V_\infty = (V_\infty, +, \cdot)$ on its open $c$-ball $V_c$. Then, the Poincaré distance function* (4.1) *on $V_c$ satisfies the inequality*

$$d(\mathbf{x}, \mathbf{y}) \le d(\mathbf{x}, \mathbf{z}) \oplus d(\mathbf{y}, \mathbf{z}) \le d(\mathbf{x}, \mathbf{z}) + d(\mathbf{y}, \mathbf{z}).$$

*Proof.* Since gyroautomorphisms are norm preserving operators we have, by Lemma 4.1,

$$\begin{aligned} \|\mathbf{x} \ominus \mathbf{y}\| &= \|\mathrm{gyr}[-\mathbf{z}; \mathbf{x}](\mathbf{x} \ominus \mathbf{y})\| \\ &= \|(-\mathbf{z} \oplus \mathbf{x}) \ominus (-\mathbf{z} \ominus \mathbf{y})\|. \end{aligned}$$

We continue this chain of equations noting that, in $V_c$, $\mathbf{a} \ominus \mathbf{b}$ denotes $\mathbf{a} \oplus (-\mathbf{b})$; and that according to [15, Theorem 5.2] we have $-(\mathbf{a} \oplus \mathbf{b}) = -\mathbf{a} \ominus \mathbf{b}$. Hence,

$$\begin{aligned} &= \|(-\mathbf{z} \oplus \mathbf{x}) \oplus (\mathbf{z} \ominus \mathbf{y})\| \\ &\le \|-\mathbf{z} \oplus \mathbf{x}\| \oplus \|\mathbf{z} \ominus \mathbf{y}\| && \text{by Theorem 4.1} \\ &= d(\mathbf{x}, \mathbf{z}) \oplus d(\mathbf{y}, \mathbf{z}) \\ &\le d(\mathbf{x}, \mathbf{z}) + d(\mathbf{y}, \mathbf{z}) && \text{by Theorem 4.2.} \end{aligned}$$

The proof is thus complete.

DEFINITION 4.2 (Gyrotranslations and Rotations of $V_c$). Let $V_\infty = (V_\infty, +, \cdot)$ be a real inner product space, let $V_c = (V_c, \oplus, \mathrm{gyr})$ be the gyrogroup induced by $V_\infty$ on its open $c$-ball $V_c$, and let $\mathrm{Aut}(V_\infty) = \mathrm{Aut}(V_\infty, +, \cdot)$ be the group of all automorphisms of $V_\infty$ (that is, all bijections of $V_\infty$ that respect its binary operation and preserve its inner product). Viewing elements $(\mathbf{a}, A)$ of the gyrosemidirect product group $V_c \times_g \mathrm{Aut}(V_\infty)$ as *holomorphic automorphisms* of $V_c$,

$$(\mathbf{a}, A)\mathbf{x} = \mathbf{a} \oplus A\mathbf{x}. \tag{4.4}$$

(The automorphisms $A: V_\infty \to V_\infty$ preserve the inner product in $V_\infty$, by definition, so that $A$ takes $V_c$ into itself.) We call, respectively, the holomorphic automorphisms $(\mathbf{a}, I)$ and the holomorphic automorphisms $(\mathbf{0}, A)$ of $V_c$ *gyrotranslations* and *rotations*. Here $\mathbf{0}$ is the zero vector in $V_\infty$, and $I$ is the identity automorphism of $V_c$.

Clearly, the rotations $(\mathbf{0}, A)$ of $V_c$ form a group isomorphic with $\mathrm{Aut}(V_\infty, +, \cdot)$, and the gyrotranslations $(\mathbf{a}, I)$ of $V_c$ form a gyrogroup isomorphic (in some sense; see [23]) with the gyrogroup $(V_c, \oplus)$. In this sense we say that the holomorphic automorphisms of Definition 4.2 form the *gyrosemidirect product* group

$$V_c \times_g \mathrm{Aut}(V_\infty) \tag{4.5}$$

of $V_c$ and $\mathrm{Aut}(V_\infty)$.

The product of any two elements of the gyrosemidirect product group (4.5) is given by automorphism composition: For any $(\mathbf{a}, A), (\mathbf{b}, B) \in V_c \times_g \mathrm{Aut}(V_\infty)$ we have the product (called *gyrosemidirect* product)

$$(\mathbf{a}, A)(\mathbf{b}, B) = (\mathbf{a} \oplus A\mathbf{b}, \mathrm{gyr}[\mathbf{a}; A\mathbf{b}]AB) \tag{4.6}$$

which is a group operation since it is derived from automorphism composition. Viewing $(\mathbf{a}, A)$ and $(\mathbf{b}, B)$ as automorphisms of $V_c$ according to Eq. (4.4), and employing the right gyroassociative law, we have for all $\mathbf{x} \in V_c$ the automorphism composition

$$\begin{aligned}
(\mathbf{a}, A)(\mathbf{b}, B)\mathbf{x} &= (\mathbf{a}, A)(\mathbf{b} \oplus B\mathbf{x}) \\
&= \mathbf{a} \oplus A(\mathbf{b} \oplus B\mathbf{x}) \\
&= \mathbf{a} \oplus (A\mathbf{b} \oplus AB\mathbf{x}) \\
&= (\mathbf{a} \oplus A\mathbf{b}) \oplus \mathrm{gyr}[\mathbf{a}; A\mathbf{b}]AB\mathbf{x} \\
&= (\mathbf{a} \oplus A\mathbf{b}, \mathrm{gyr}[\mathbf{a}; A\mathbf{b}]AB)\mathbf{x}.
\end{aligned}$$

The gyrosemidirect product is known in the literature; Kikkawa [10] calls it (ambiguously) a semidirect product. Clearly, our gyrosemidirect product reduces to the standard semidirect product of group theory in the special case when all gyrations vanish (a case in which gyrogroups specialize in groups).

THEOREM 4.4. *Let $V_c = (V_c, \oplus, \mathrm{gyr})$ be the gyrogroup induced by a real inner product space $V_\infty = (V_\infty, +, \cdot)$ on its open c-ball $V_c$. Then, the gyrotranslations and rotations of $V_c$ (that is, the holomorphic automorphisms of $V_c$ in Definition* 4.2) *preserve its Poincaré metric.*

*Proof.* The holomorphic automorphisms $(\mathbf{a}, A)$ of $V_c$,

$$(\mathbf{a}, A) \in V_c \times_g \operatorname{Aut}(V_\infty),$$

form the (gyrosemidirect product) group of all gyrotranslations and rotations of $V_c$. We have to show that these preserve the Poincaré distance function $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} \ominus \mathbf{y}\|$ on $V_c$. Let $\mathbf{x}, \mathbf{y} \in V_c$ and let

$$\mathbf{x}' = (\mathbf{a}, A)\mathbf{x} = \mathbf{a} \oplus A\mathbf{x}$$

$$\mathbf{y}' = (\mathbf{a}, A)\mathbf{y} = \mathbf{a} \oplus A\mathbf{y}.$$

Then, by Lemma 4.1, and since automorphisms $(\mathbf{0}, A)$ (and, in particular, gyroautomorphisms) of $V_c$ preserve the norm that elements of $V_c$ inherit from $V_\infty$, we have

$$\begin{aligned}
\|\mathbf{x}' \ominus \mathbf{y}'\| &= \|(\mathbf{a} \oplus A\mathbf{x}) \ominus (\mathbf{a} \oplus A\mathbf{y})\| \\
&= \|\operatorname{gyr}[\mathbf{a}; A\mathbf{x}](A\mathbf{x} \ominus A\mathbf{y})\| \\
&= \|\operatorname{gyr}[\mathbf{a}; A\mathbf{x}]A(\mathbf{x} \ominus \mathbf{y})\| \\
&= \|\mathbf{x} \ominus \mathbf{y}\|
\end{aligned}$$

and the proof is complete.

## 5. SOME VECTORLIKE FEATURES OF THE GYROGROUP INDUCED BY A REAL VECTOR SPACE

Elements of the gyrogroup $V_c$ are vectors in the real inner product space $V_\infty$. It is therefore interesting to realize that the gyrogroup $V_c$ possesses some vectorlike features. We define the *scalar multiplication* $r \odot \mathbf{x}$ of $\mathbf{x} \in V_c$ by a real number $r \in \mathbb{R}$ by the equation

$$r \odot \mathbf{x} = \frac{c}{\|\mathbf{x}\|} \frac{(c + \|\mathbf{x}\|)^r - (c - \|\mathbf{x}\|)^r}{(c + \|\mathbf{x}\|)^r + (c - \|\mathbf{x}\|)^r} \mathbf{x}, \tag{5.1a}$$

for $\|\mathbf{x}\| \neq 0$, and by the equation $r \odot \mathbf{0} = \mathbf{0}$ otherwise. Equivalently, the scalar multiplication in Eq. (5.1a) can be written as

$$r \odot \mathbf{x} = \frac{c}{\|\mathbf{x}\|} \tanh\left( r \tanh^{-1} \frac{\|\mathbf{x}\|}{c} \right) \mathbf{x}. \tag{5.1b}$$

In can readily be shown that $r \odot \mathbf{x} \in V_c$ for any $\mathbf{x} \in V_c$ and $r \in \mathbb{R}$, and that $r \odot \mathbf{x}$ goes over to $r\mathbf{x}$ when $c \to \infty$,

$$\lim_{c \to \infty} r \odot \mathbf{x} = r\mathbf{x}$$

for any $\mathbf{x} \in V_\infty$ where $r\mathbf{x}$ is the usual product of a vector by a scalar in the real vector space $V_\infty$. In the special case when $n$ is a positive integer we have

$$n \odot \mathbf{x} = \mathbf{x} \oplus \mathbf{x} \oplus \cdots \oplus \mathbf{x} \qquad (\text{gyroadding } n \text{ times}).$$

Furthermore, for any two real numbers $r_1$ and $r_2$ and for any $\mathbf{x} \in V_c$ we have

$$(r_1 + r_2) \odot \mathbf{x} = (r_1 \odot \mathbf{x}) \oplus (r_2 \odot \mathbf{x})$$

and

$$(r_1 r_2) \odot \mathbf{x} = r_1 \odot (r_2 \odot \mathbf{x}).$$

Unfortunately, a distributive-like law connecting the two binary operations $\oplus$ and $\odot$ in $V_c$ is unknown; in general, we have the inequality

$$r \odot (\mathbf{x} \oplus \mathbf{y}) \neq (r \odot \mathbf{x}) \oplus (r \odot \mathbf{y}).$$

Interestingly, the norm $\|\mathbf{x}\|$ of $\mathbf{x}$ in the vector space $V_\infty$ acts as a norm of $\mathbf{x}$ in the gyrogroup $V_c \subset V_\infty$ as well in the sense that

$$\|r \odot \mathbf{x}\| = |r| \odot \|\mathbf{x}\|.$$

## 6. INVARIANTS OF THE GENERALIZED MÖBIUS TRANSFORMATION

The binary operation $\oplus$ in $V_c$, Eq. (3.15), is a most natural generalization of the binary operation $\oplus$ in $D_c$, Eq. (2.1). The latter, in turn, is recognized as a Möbius transformation commonly denoted in the literature by $\phi_a$. Thus, in $D_{c=1}$,

$$a \oplus z = \frac{a + z}{1 + \bar{a}z} = -\phi_a(-z), \tag{6.1}$$

where (see, e.g., [2, p. 133] for $D_{c=1}$ and [13, p. 25; 20] for $V_{c=1}$)

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z}. \tag{6.2}$$

By Definition 4.2, our holomorphic automorphisms of $V_c$ are thus the generalized pure Möbius transformations $\phi_a$ and the rotations of $V_c$. Möbius transformations of $D_c$ preserve the *cross ratio* in the extended complex plane $\mathbb{C} \cup \infty$. Accordingly, we wish to discover in this section invariants of the generalized Möbius transformations.

DEFINITION 6.1. The *scalar cross ratio* of any four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in (V_\infty, +, \cdot)$ is the real number or $\infty$ given by

$$\frac{\|\mathbf{a}-\mathbf{c}\|}{\|\mathbf{a}-\mathbf{d}\|} \frac{\|\mathbf{b}-\mathbf{d}\|}{\|\mathbf{b}-\mathbf{c}\|}. \tag{6.3}$$

THEOREM 6.1. *Excluding singularities, the generalized Möbius transformations of* $(V_c, \oplus)$ *acting on* $(V_\infty, +, \cdot)$ *keep the scalar cross ratio of any four vectors in* $V_\infty$ *invariant.*

*Proof.* The scalar cross ratio is clearly invariant under rotations of $V_\infty$. By straightforward computer algebra one can see that it is also invariant under left gyrotranslations of $V_\infty$.

DEFINITION 6.2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in (V_\infty, +, \cdot)$ be any four vectors in $V_\infty$, and let

$$\begin{aligned} A &= \mathbf{a} - \mathbf{c} \\ B &= \mathbf{b} - \mathbf{d} \\ C &= \mathbf{a} - \mathbf{d} \\ D &= \mathbf{b} - \mathbf{c}. \end{aligned} \tag{6.4}$$

The angle associated with $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ is

$$\frac{-(A \cdot B)(C \cdot D) + (A \cdot C)(B \cdot D) + (A \cdot D)(B \cdot C)}{\|A\| \, \|B\| \, \|C\| \, \|D\|}. \tag{6.5}$$

THEOREM 6.2. *Excluding singularities, the generalized Möbius transformations of* $V_\infty$ *keep the associated angle* (6.5) *invariant for any* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in V_\infty$.

*Proof.* The generalized angle is clearly invariant under rotations of $V_\infty$ about its origin. By straightforward computer algebra one can see that it is also invariant under generalized pure Möbius transformations.

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## REFERENCES

1. L. V. Ahlfors, Cross ratios and Schwarzian derivatives in $\mathbb{R}^n$, *in* "Complex Analysis Articles Dedicated to Albert Pfluger on the Occasion of his 80th Birthday" (J. Hersch and A. Huber, Eds.), pp. 1–15, Birkhäser, Basel, 1988.
2. C. A. Berenstein and R. Gay, "Complex Variables An Introduction," Springer-Verlag, New York, 1991.

3. H. S. Bear, Distance-decreasing functions on the hyperbolic plane, *Michigan Math. J.* **39** (1992), 271–279.
4. B. R. Ebanks and C. T. Ng, On generalized cocycle equations, *Aequationes Math.* **46** (1993), 76–90.
5. A. Dittmer, Cross product identities in arbitrary dimension, *Amer. Math. Monthly* **9** (1994), 887–891.
6. K. Goebel and S. Reich, ‘‘Uniform Convexity Hyperbolic Geometry and Nonexpansive Mappings,’’ pp. 97–98, Dekker, New York, 1984.
7. P. R. Halmos, ‘‘Finite Dimensional Vector Spaces,’’ Van Nostrand, New York, 1958.
8. S. H. Hochwald, Linear algebra by analogy, *Amer. Math. Monthly* **98** (1991), 918–926.
9. J. M. Isidro and L. L. Stachó, ‘‘Holomorphic Automorphism Groups in Banach Spaces: An Elementary Introduction’’ (L. Nachbin, Ed.), North-Holland Mathematics Studies, Vol. 105; Notas Math., Vol. 97, North-Holland, New York, 1984.
10. M. Kikkawa, Geometry of homogeneous Lie loops (Section 2), *Hiroshima Math. J.* **5** (1975), 141–179.
11. S. Kobayashi, ‘‘Hyperbolic Manifolds and Holomorphic Mappings,’’ Dekker, New York, 1970.
12. J. Maks, A group-like structure on the unit ball in $\mathbb{R}^n$, preprint.
13. W. Rudin, ‘‘Function Theory in the Unit Ball of $\mathbb{C}^n$,’’ Springer-Verlag, New York, 1980.
14. J. D. H. Smith and A. A. Ungar, Abstract space-times and their Lorentz groups, *J. Math. Phys.* **37** (1996), 3073–3098.
15. A. A. Ungar, Weakly associative groups, *Res. Math.* **17** (1990), 149–168.
16. A. A. Ungar, The expanding Minkowski space, *Res. Math.* **17** (1990), 342–354.
17. A. A. Ungar, Thomas precession and its associated grouplike structure, *Amer. J. Phys.* **59** (1991), 824–834.
18. A. A. Ungar, Solving the velocity composition equation of special relativity, Problem 6659, *Amer. Math. Monthly* **98** (1991), 445–446; and its solution, *Amer. Math. Monthly* **100** (1993), 500–502.
19. A. A. Ungar, The holomorphic automorphism group of the complex disk, *Aequationes Math.* **47** (1994), 240–254.
20. A. A. Ungar, The abstract complex Lorentz transformation group with real metric. I. Special relativity formalism to deal with the holomorphic automorphism group of the unit ball in any complex Hilbert space, *J. Math. Phys.* **35** (1994), 1408–1425.
21. A. A. Ungar, Erratum: ‘‘The abstract complex Lorentz transformation group with real metric. I. Special relativity formalism to deal with the holomorphic automorphism group of the unit ball in any complex Hilbert space,’’ *J. Math. Phys.* **35** (1994), 3770.
22. A. A. Ungar, The abstract complex Lorentz transformation group with real metric. II. The invariance group of the form $|t|^2 - \|\mathbf{x}\|^2$, *J. Math. Phys.* **35** (1994), 1881–1913.
23. Y. You and A. A. Ungar, Equivalence of two gyrogroup structures on unit balls, *Res. Math.* **28** (1995), 359–371.